The wave function of a particle constraint to move along x $(-\infty < x < +\infty)$ at a certain instant is given by $\psi(x) = Ae^{-x^2/a^2 + ibx}$ (where a and b are real constants). Find the normalization constant A.

$\Rightarrow$ The condition of normalization is

$$\int_{-\infty}^{+\infty} \psi^*(x)\, \psi(x)\, dx = 1$$

But, $\psi(x) = Ae^{-\frac{x^2}{a^2} + ibx}$

& $\psi^*(x) = A^* e^{-\frac{x^2}{a^2} - ibx}$

Now, $$\int_{-\infty}^{+\infty} |A|^2 e^{-\frac{x^2}{a^2} + ibx} \times e^{-\frac{x^2}{a^2} - ibx}\, dx = 1$$

$$\Rightarrow A^2 \int_{-\infty}^{\infty} e^{-2\frac{x^2}{a^2}}\, dx = 1$$

Let, $\frac{2x^2}{a^2} = z$

$\Rightarrow x^2 = \frac{a^2 z}{2}$

$\therefore x = \frac{a\sqrt{z}}{\sqrt{2}}$

$\therefore 2x\, dx = \frac{a^2}{2}\, dz$

$dx = \frac{a^2}{4}\frac{dz}{x} = \frac{a}{2\sqrt{2}} z^{-1/2}\, dz$

$$\therefore A^2 \int_{-\infty}^{+\infty} e^{-z} \frac{a}{2\sqrt{2}} z^{-1/2}\, dz = 1$$

$$\Rightarrow A^2 \frac{a}{2\sqrt{2}} \int_{-\infty}^{+\infty} e^{-z} z^{\frac{1}{2}-1}\, dz = 1$$

$\Rightarrow A^2 \frac{a}{2\sqrt{2}} \Gamma\frac{1}{2} = 1 \qquad \therefore A = \left(\frac{4}{\sqrt{\pi}\, a}\right)^{1/2}$

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Show that $[x^n, \hat{p}_x] = i\hbar n x^{n-1}$, n is a positive integer.

$\Rightarrow$ If ψ is a state function,

Then, $\hat{p}_x x^n \psi = (-i\hbar)\frac{\partial}{\partial x}(x^n \psi)$

$= (-i\hbar) n x^{n-1} \psi + (-i\hbar) x^n \frac{\partial \psi}{\partial x}$

$\Rightarrow [x^n \hat{p}_x - \hat{p}_x x^n]\psi = (-i\hbar) n x^{n-1}$

$\therefore \boxed{[x^n, \hat{p}_x] = (i\hbar) n x^{n-1}}$

Calculate the value of $\langle r \rangle$ related to the wave function $\psi(r) = \sqrt{\frac{1}{\pi a^3}}\, e^{-r/a}$

$\Rightarrow$ We have expectation value of r

$\langle r \rangle = \int \psi^* \hat{r} \psi \, d\tau$

$= \int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} (\psi^* \hat{r} \psi)\, d\phi$

$\because d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$= (\sqrt{\pi a^3})^2 \times 4\pi \int_0^\infty r^3 e^{-2r/a} dr$

$= (\pi a^3) \times 4\pi \int_0^\infty e^{-2r/a} r^3 dr$

$= \pi a^3 \times 4\pi \times \frac{3}{8\pi^2 a^2}$

$= \frac{3}{2} a$

Calculate $\langle p \rangle$ and $\langle p^2 \rangle$ for the wave function

$$\psi(x) = \begin{cases} \left(\frac{2}{L}\right)^{1/2} \sin\frac{\pi x}{L} & 0 < |x| < L \\ 0 & |x| > L \end{cases}$$

$$\Rightarrow \langle P \rangle = \frac{\int_{-\infty}^{+\infty} \psi^*(x)\, \hat{P}\, \psi(x)\, dx}{\int_{-\infty}^{+\infty} \psi^*(x)\psi(x)\, dx}$$

$$= \frac{\frac{1}{2}\int_{-L}^{+L} \left(\frac{2}{L}\right) \sin\frac{\pi x}{L}\left(-i\hbar\frac{\partial}{\partial x}\right)\sin\frac{\pi x}{L}\, dx}{\int_{-L}^{+L} \psi^*(x)\psi(x)\, dx}$$

$$= \frac{1}{2}\int_{-L}^{+L}\left(\frac{2}{L}\right)\sin\frac{\pi x}{L}(-i\hbar)\frac{\pi}{L}\cos\frac{\pi x}{L}\, dx$$

$$= \frac{\pi\hbar}{2iL^2}\int_{-L}^{+L} 2\sin\frac{\pi x}{L}\cos\frac{\pi x}{L}\, dx$$

$$= \frac{\pi\hbar}{2iL^2}\int_{-L}^{+L}\sin\frac{2\pi x}{L}\, dx$$

$$= 0 \qquad \Big| \because \sin\frac{2\pi x}{L} \text{ is an odd function of } x$$

$$\boxed{\therefore \langle P \rangle = 0}$$

again, $$\langle p^2 \rangle = \frac{\int_{-\infty}^{+\infty}\psi^*(x)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\psi(x)\, dx}{\int_{-\infty}^{+\infty}\psi^*(x)\psi(x)\, dx}$$

$$= \frac{\hbar^2}{2}\frac{2}{L}\left(\frac{\pi}{L}\right)^2\int_{-L}^{+L}\sin\frac{\pi x}{L}\sin\frac{\pi x}{L}\, dx$$

$$\langle p^2 \rangle = \frac{\pi^2 \hbar^2}{2L^3} \int_{-L}^{+L} 2 \sin^2 \frac{\pi x}{L} dx$$

$$= \frac{\pi^2 \hbar^2}{2L^3} \int_{-L}^{+L} \left(1 - \cos \frac{2\pi x}{L}\right) dx$$

$$= \frac{\pi^2 \hbar^2}{2L^3} \left[x - \frac{\sin \frac{2\pi x}{L}}{\frac{2\pi}{L}} \right]_{-L}^{+L}$$

$$= \frac{\pi^2 \hbar^2}{2L^3} \left[2L - \frac{L}{2\pi} (0-0) \right]$$

$$\boxed{\therefore \langle p^2 \rangle = \frac{\pi^2 \hbar^2}{L^2}}$$

Show that the momentum operator $\frac{\hbar}{i}\frac{\partial}{\partial x}$ is hermitian.

$\Rightarrow$ The momentum operator

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

complex conjugate of $\hat{p}$ is

$$\hat{p}^* = -\frac{\hbar}{i} \frac{\partial}{\partial x}$$

If $\hat{p}$ is Hermition operators, then in a given state its expectation value would be real

i.e

$$\langle \hat{p} \rangle = \int_{-\infty}^{+\infty} \psi^* \frac{\hbar}{i} \frac{\partial \psi}{\partial x} dx$$

Integrating by parts, we get

$$\langle \hat{p} \rangle = \frac{\hbar}{i}\left[\psi^* \psi\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{\hbar}{2} \frac{\partial \psi^*}{\partial x} \psi \, dx$$

$$= \int_{-\infty}^{+\infty} \psi \left(-\frac{\hbar}{i} \frac{\partial \psi^*}{\partial x} \right) dx$$

$$= \langle \hat{p}^* \rangle$$

Since $\langle \hat{p} \rangle = \langle \hat{p}^* \rangle$, $\langle \hat{p} \rangle$ is real and $\hat{p}$ is Hermitian.

\# Prove that if two eigen functions operated by the same Hamiltonian operator give two eigen values, then they will be orthogonal.

$\Rightarrow$ If ψ_m and ψ_n be two eigen functions corresponding to eigen values E_m and E_n of the schrödinger equation then in one dimension case.

$$\int_{-\infty}^{+\infty} \psi_n^*(x)\, \psi_m(x)\, dx = 0 \quad \text{if } E_m \neq E_n \quad \text{—— (1)}$$

$$\text{and} \int_{-\infty}^{+\infty} \psi_n^*(x)\, \psi_m(x)\, dx = 1 \quad \text{if } E_m = E_n \quad \text{—— (2)}$$

The eqn (2) represents that ψ_m is normalized

Now time independent schrödinger eqn for ψ_m in one dimension is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_m(x)}{dx} + V(x)\,\psi_m(x) = E_m\,\psi_m(x) \qquad \text{— (3)}$$

For ψ_n^*

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n^*(x)}{dx^2} + V(x)\,\psi_n^*(x) = E_n\,\psi_n^*(x) \qquad \text{— (4)}$$

Multiplying eqn (3) ψ_n^* and eqn (4) by ψ_m and then subtracting and integrating from $x = -\infty$ to $+\infty$ we get

$$-\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left[\psi_n^*\frac{d^2\psi_m}{dx^2} - \psi_m\frac{d^2\psi_n^*}{dx^2}\right]dx = (E_m - E_n)\int_{-\infty}^{+\infty}\psi_n^*\,\psi_m\,dx$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[\psi_n^*\frac{d\psi_m}{dx} - \psi_m\frac{d\psi_n}{dx}\right]_{-\infty}^{+\infty} = (E_m - E_n)\int_{-\infty}^{+\infty}\psi_n^*\,\psi_m\,dx$$

ψ and $\frac{d\psi}{dx} \to 0$ as $x \to \pm\infty$

$$\therefore (E_m - E_n)\int_{-\infty}^{+\infty}\psi_n^*\,\psi_m\,dx = 0$$

but $E_m \neq E_n$

$$\therefore \int_{-\infty}^{+\infty}\psi_n^*\,\psi_m\,dx = 0$$

that is ψ_m and ψ_n are orthogonal.